

Cosmological quantum tunneling and dimension of the universe

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Abstract

The probable trajectory of the ground state wave function of the universe arises through a quantum tunneling by gravitational instantons. We calculate the quantum tunneling rate for an n -dimensional closed Friedmann-Robertson-Walker universe with a cosmological constant. It is shown that the highest rate for creation of a Planck scale de Sitter universe from nothing occurs when the dimension of space-time is very close to four.

Keywords: Quantum cosmology, Tunneling rate, space-time dimensionality.

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1 Introduction

Interest in explaining why and how the space-time has become a $4D$ manifold is not a new challenge of human mind. In fact, most of explanations made to date are essentially based on the anthropic principle [1]. Our existence in the universe is simply considered as an anthropic reason for four dimensionality of the space-time. If space would have more than three dimensions, no stable bound orbits of planets around the sun would be possible and so no human kind would exist. Also in quantum mechanics the same argument applies to the existence of stable atoms which are the building blocks of every material structure. Trying to explain the dimensionality of space-time without resorting to the anthropic principle is an important problem which is usually supposed to be addressed within the fundamental theories like string or supersrting theory. In such fundamental theories, one assumes an n -dimensional space-time and by demanding some special consistency requirements in the theory the number of dimensions is obtained [2]. However, a priori, it is usually assumed that four, out of n dimensions, are large ordinary space-time and the remaining ones are compact space of the Planck scale. An interesting attempt to drive dynamically a $4D$ space-time in string theory has been made in the non-perturbative \mathbf{M} formulation of type IIB strings [3]. On the other hand, there are some approaches with special interest in finding a reasonable answer and justification for the question that why four dimensional space-time has Lorentzian signature [4], why nature has made a choice of one time and three space coordinates [5], or why do we live in $3+1$ dimensions [6].

In general relativity, our base assumption is that space-time in large scales is a $4D$ differentiable manifold. On the other hand, in small scales like the Planck length, quantum effects will be important and space-time will be highly curved with all possible topologies and of arbitrary dimensions. So, the main question is: how a 4-dimensional space-time is singled out of other quantum mechanically possible configurations. In the present paper, we aim to answer this question by limiting ourselves to the dimensionality, rather than topology of space-time. To this end, we consider an n -dimensional empty closed ($k = 1$) Friedmann-Robertson-Walker (FRW) universe with a cosmological constant and then calculate the tunneling rate for creation of this universe from nothing. It turns out that the tunneling rate depends on the number of dimensions, and so the maximum of this rate fixes the dimensions of space-time to be very close to four, in agreement with reality.

2 The model

We consider an n -dimensional empty closed ($k = 1$) Friedmann-Robertson-Walker (FRW) universe with a cosmological constant Λ . The line element is given by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-r^2} + r^2 d\Omega_{n-2}^2 \right], \quad (1)$$

where $a(t)$ as the scale factor is the only dynamical degree of freedom and

$$d\Omega_{n-2}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-3} d\theta_{n-2}^2. \quad (2)$$

The spatial measure over the space-like hypersurface \mathcal{V} corresponding to this metric is obtained

$$\begin{aligned} V &= \int_{\mathcal{V}} \sqrt{-g} dr d\theta_1 \dots d\theta_{n-3} \\ &= a^{n-1}(t) \int_0^r \frac{r^{n-2} dr}{\sqrt{1-r^2}} \int_0^\pi (\sin \theta_1)^{n-3} d\theta_1 \dots \int_0^\pi \sin \theta_{n-3} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2}. \end{aligned} \quad (3)$$

Using the following integral

$$\int_0^r \frac{r^{n-2} dr}{\sqrt{1-r^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}, \quad (4)$$

we obtain the measure as follows

$$\int_{\mathcal{V}} \sqrt{-g} dr d\theta_1 \dots d\theta_{n-3} = a^{n-1}(t) \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \quad (5)$$

The Ricci scalar corresponding to the metric (1) is given by

$$\mathcal{R} = (n-1) \left[2\frac{\ddot{a}}{a} + (n-2)\frac{\dot{a}^2}{a^2} + (n-2)\frac{1}{a^2} \right], \quad (6)$$

where, for $n = 4$, we recover the standard 4-dimensional Ricci scalar

$$\mathcal{R} = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \right]. \quad (7)$$

The gravitational action corresponding to this model is given by

$$\begin{aligned} S &= \frac{1}{16\pi G_n} \int dt (\mathcal{R} - 2\Lambda) \int_{\mathcal{V}} \sqrt{-g} dr d\theta_1 \dots d\theta_{n-3} \\ &= \frac{1}{16\pi G_n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int dt (n-1) a^{n-1}(t) \left[2\frac{\ddot{a}}{a} + (n-2)\frac{\dot{a}^2}{a^2} + (n-2)\frac{1}{a^2} \right] \\ &= M_P^{n-2} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} (n-1) \int dt [2a^{n-2}\ddot{a} + (n-2)a^{n-3}\dot{a}^2 + (n-2)a^{n-3}], \end{aligned} \quad (8)$$

where G_n is the n-dimensional gravitational constant and the Planck constant in n-dimension is $M_P^{n-2} = 1/16\pi G_n$. After a total derivative on the term containing \ddot{a} , the final form of the action is obtained

$$S = -(n-1)(n-2)M_P^{n-2} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int dt \left[a^{n-3}(\dot{a}^2 - 1) + \frac{2\Lambda}{(n-1)(n-2)} a^{n-1} \right]. \quad (9)$$

Therefore, the Lagrangian is given by

$$\mathcal{L} = -\lambda \left[a^{n-3}(\dot{a}^2 - 1) + \frac{2\Lambda}{(n-1)(n-2)} a^{n-1} \right], \quad (10)$$

where

$$\lambda = (n-1)(n-2)M_P^{n-2} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \quad (11)$$

The momentum conjugate to the scale factor is given by

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -2\lambda \dot{a} a^{n-3}. \quad (12)$$

Then, we obtain the Hamiltonian as

$$\mathcal{H} = \dot{a}\Pi_a - \mathcal{L} = -\frac{1}{4\lambda a^{n-3}}\pi_a^2 - \lambda a^{n-3} + \frac{2\Lambda\lambda}{(n-1)(n-2)}a^{n-1}. \quad (13)$$

The zero energy requirement of this gravitational system leads to the following Hamiltonian constraint

$$\Pi_a^2 + 4\lambda^2 a_0^{2n-6} q^{2n-6} [1 - q^2] = 0, \quad (14)$$

where $q = a/a_0$ and

$$a_0^2 = \frac{(n-1)(n-2)}{2\Lambda}. \quad (15)$$

3 Classical Cosmology

Using (15), the Lagrangian (10) is rewritten as

$$\mathcal{L} = -\lambda \left[a^{n-3} (\dot{a}^2 - 1) + \frac{1}{a_0^2} a^{n-1} \right]. \quad (16)$$

The Euler-Lagrange equation is then obtained

$$\frac{\ddot{a}}{a} + \frac{n-3}{2} \frac{(\dot{a}^2 - 1)}{a^2} - \frac{n-1}{2a_0^2} = 0. \quad (17)$$

The Hamiltonian constraint (14) results in the following equation

$$H^2 + \frac{1}{a^2} - \frac{1}{a_0^2} = 0, \quad (18)$$

where $H = \dot{a}/a$ is the Hubble parameter. Combining Eqs.(17), (18) we obtain

$$\frac{\ddot{a}}{a} - \frac{1}{a_0^2} = 0. \quad (19)$$

The solution of this equation, subject to the Hamiltonian constraint (18), is obtained

$$a(t) = a_0 \cosh(t/a_0), \quad (20)$$

where a_0 is interpreted as the minimum radius of the universe and the initial conditions $a(0) = a_0$ and $\dot{a}(0) = 0$ are used. This solution corresponds to a usual de Sitter spacetime where a phase of contraction from infinitely past time is followed by an expansion phase where the scale factor has reached its minimum a_0 . However, a different cosmological scenario can give rise to an identical de Sitter expansion if we consider analytic continuation $\tau = it + (\pi/2)a_0$. Then, one can obtain the instanton solution as

$$a_E = a_0 \sin\left(\frac{\tau}{a_0}\right), \quad (21)$$

so that the instanton is an n -sphere of radius a_0 if $\tau/a_0 \in \{-\pi/2, \pi/2\}$, with spherical three-dimensional sections labelled by the latitude angle $\theta = a_0\tau$. Both metrics are related by the analytic continuation into the complex plane of the Euclidean “time” τ

$$\tau = \frac{\pi}{2}a_0 + it, \quad a = a_E\left(\frac{\pi}{2}a_0 + it\right), \quad (22)$$

which is a Wick rotation with respect to the point $\tau = (\pi/2)a_0$ in this plane. This analytic continuation can be interpreted as a quantum nucleation of the Lorentzian de Sitter spacetime from the Euclidean hemisphere as a matching of the two manifolds across the equatorial section $\tau = (\pi/2)a_0$, ($t = 0$), the bounce surface of zero extrinsic curvature.

4 Quantum Cosmology

Canonical quantization of this cosmological model in the coordinate representation is accomplished by the operator realizations

$$a = a, \quad \Pi_a = -i\hbar \frac{\partial}{\partial a}. \quad (23)$$

Then, the Hamiltonian constraint (14) becomes the Wheeler-DeWitt equation for the wave function of the universe

$$\frac{d^2\psi}{dq^2} + Q^2\psi = 0, \quad (24)$$

where $Q^2 = 4\lambda^2 a_0^{2(n-2)} q^{2(n-3)} [q^2 - 1]$, and use has been made of $q = a/a_0$ to convert the dynamical variable from a to q . The solutions of the Wheeler-DeWitt equation are obtained as

$$\psi = \begin{cases} \frac{N}{2}|Q|^{-1/2} \exp|w|, & q < 1 \\ N|Q|^{-1/2} \cos(|w| - \pi/4), & q > 1 \end{cases} \quad (25)$$

where

$$w = \int^q Q dq. \quad (26)$$

The tunnelling rate in WKB approximation is then obtained

$$|T|^2 = \left[1 + \exp\left(2i \int_0^1 Q dq\right) \right]^{-1}, \quad (27)$$

where the integral in the argument is calculated as follows

$$\begin{aligned} \int_0^1 Q dq &= 2\lambda a_0^{n-2} \int_0^1 q^{n-3} \sqrt{q^2 - 1} dq \\ &= i \frac{\lambda}{2} a_0^{n-2} \sqrt{\pi} \frac{\Gamma(n/2 - 1)}{\Gamma(n/2 + 1/2)}. \end{aligned} \quad (28)$$

Therefore, we have

$$|T|^2 = \left[1 + \exp\left(-\lambda a_0^{n-2} \sqrt{\pi} \frac{\Gamma(n/2 - 1)}{\Gamma(n/2 + 1/2)}\right) \right]^{-1}, \quad (29)$$

Substituting (11) into (29), and after some calculations, we obtain the final result

$$|T|^2 = [1 + \exp(-\alpha)]^{-1}, \quad (30)$$

where

$$\alpha = \frac{4\pi^{3/2}}{\Gamma(n/2 - 1/2)} [\pi a_0^2 G^{-1}]^{n/2-1}. \quad (31)$$

This formula shows the dependence of tunneling rate on the dimension of space-time n , the gravitational constant G , and also the minimum radius of the classical universe after quantum tunneling, namely a_0 . The tunnelling rate becomes considerable for larger values of α . For $n = 1$, the tunnelling rate is not defined. For $n = 2$, the tunnelling rate is a constant

$$|T|^2 = [1 + \exp(-4\pi)]^{-1}. \quad (32)$$

For $n \geq 3$ the tunnelling rate depends on the parameters n , G and a_0 . It is interesting to investigate how the tunnelling rate gets a maximum. To this end, we shall assume

$$a_0 = l_P / \pi, \quad (33)$$

where l_P stands for the Planck length. This assumption is based on the common belief that the classical universe should start its existence at the Planck scale. In the units $\hbar = c = 1$ we have $\pi a_0^2 G^{-1} = l_P^2 M_P^2 = 1$. Therefore, we end up with the following expression

$$\alpha = \frac{4\pi^{3/2}}{\Gamma(n/2 - 1/2)}. \quad (34)$$

Now, this quantity has a maximum very close to $n=4$ which is a very interesting result. In other words, the quantum cosmological considerations on the cosmological model at hand asserts that the highest tunneling rate for creation of a Planck scale de Sitter universe from nothing occurs when the space-time dimension approaches to four, namely $n \rightarrow 4$. Therefore, the four dimensionality of space-time seems as an inevitable outcome of quantum creation of the universe.

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